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BREAKUP OF A FREE JET OF A VISCOELASTIC FLUID

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UDC 532.525.2 + 532.135

A large number of papers has been published on the breakup of a jet of a high-viscosity Newtonian liquid flowing in a low-viscosity medium. The results of these papers show that there exist three regimes of jet breakup, depending on the flow velocities. At low velocities it breaks up under the action of capillary forces, while the long-wave axially symmetric perturbations increase most quickly.

The nature of perturbation evolution changes with increasing velocity. When the dynamic action of the medium exceeds the capillary forces the long-wave bending perturbations increase significantly more quickly than the axially symmetric ones. With further velocity increase the jet breakup into large parts is changed by spraying into a set of small droplets, the size of which is independent of the jet radius. The main purpose of the present work is to determine the velocity range in which a jet of a viscoelastic liquid breaks up into large parts.

We investigate the evolution of long-wave perturbations $kR \leqslant 1$ in a circular jet of radius R of a viscoelastic liquid of density ρ, flowing with velocity U in a low-viscosity medium of density ρ_0, where k is the perturbation wave number. If for $kR \leqslant 1$ the perturbation increment increases monotonically with kR, the nature of the breakup is approximated by a spray, requiring the study of short-wave asymptotics. The analysis is based on the equations derived in [1]. We consider jets undergoing primarily extension or compression. This problem was first formulated in [2]. This mathematically insignificant complication of the problem makes it possible to estimate qualitatively the effect of a longitudinal strain, occurring in a viscoelastic jet, on its stability. We choose the rheological equation of a Maxwell liquid with viscosity η and relaxation time λ [3]: $\mathbf{T} + \lambda(D\mathbf{T}/Dt - \mathbf{W}\cdot\mathbf{T} + \mathbf{T}\cdot\mathbf{W}) + \lambda\varepsilon\cdot(\mathbf{D}\cdot\mathbf{T} + \mathbf{T}\cdot\mathbf{D}) = 2\eta\mathbf{D}$, where D/Dt is the convective derivative, $\mathbf{D}$ is the velocity deformation tensor, $\mathbf{W}$ is the vorticity tensor, $\mathbf{T}$ is the stress tensor, and ε = 0, 1, and −1, respectively, for the Jaumann, lower, and upper convective derivatives. The Maxwell liquid model is the simplest model qualitatively describing many properties of polymer liquids: instantaneous elasticity, stress relaxation, difference in normal stresses, etc. [3]. The equations for the additional capillary and hydrodynamic pressures occurring during perturbation of a jet surface $r = R + \zeta(\varphi, z, t)$ are taken from [4-6]. The system of equations for small perturbations is significantly simplified when it is not necessary to take into account the time dependence of the longitudinal stress in the jet. This assumption is valid for $t/\lambda \ll 1$. In the absence of longitudinal stress these equations describe the evolution of perturbations in a relaxing jet. A solution of the equations is sought in the form $\exp(ikz + \alpha t)$. The system of equations decomposes into separate systems, each of which corresponds to a definite perturbation. We provide the equations for the azimuthal dependence of the surface displacement and the corresponding dispersion equations. The perturbations retaining the jet linearity are:

$$\zeta = \zeta_0\, e^{\pm i2\varphi},\ \frac{1}{4}\rho\alpha^2R^2 + \frac{\alpha}{1+\alpha\lambda}\left(2\eta_1 + \eta_2\frac{k^2R^2}{4}\right) - \frac{R^2}{2}V(k, 2) = 0,$$

$$\zeta = \zeta_0\, e^{\pm i3\varphi},\ \frac{\rho\alpha^2R^2}{24} + \frac{\alpha}{1+\alpha\lambda}\left(\eta_1 + \eta_2\frac{k^2R^2}{24}\right) - \frac{R^2}{8}V(k, 3) = 0,$$

Minsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 54-63, January-February, 1982. Original article submitted December 15, 1980.

0021-8944/82/2301-0048$07.50

$$\zeta = \zeta_0 \cdot \rho\alpha^2\left(1 + \frac{k^2R^2}{8}\right) + \frac{\alpha k^2}{1+\alpha\lambda}\left(\eta_1 + 2\eta_3 + \eta_2\frac{k^2R^2}{8}\right) - \frac{Tk^2}{2} - \frac{k^2R^2}{2}V(k,0) = 0.$$

The bending perturbations $\zeta = \zeta_0 e^{\pm i\varphi}$ are:

$$\begin{aligned}
&\left[\rho\alpha^2R^2 + \frac{\alpha\eta_2k^2R^2}{1+\alpha\lambda} - V(k,1)R^2\right]\left[\left(\rho\alpha R^2\right.\right. \\
&\left. + \frac{\eta_2k^2R^2}{1+\alpha\lambda}\right)\left(1 - \frac{k^2R^2}{12}\right) + \frac{4\eta_4}{1+\alpha\lambda} \\
&\left. + \frac{2k^2R^2}{1+\alpha\lambda}\left(\eta_3 - \eta_2 - \frac{\eta_1}{2}\right)\right]\left[\frac{1}{12}\left(\rho\alpha R^2 + \frac{\eta_2k^2R^2}{1+\alpha\lambda}\right) + \frac{\eta_1}{1+\alpha\lambda}\right] \\
&+ k^2R^2\left(\rho\alpha R^2 + \frac{\eta_2k^2R^2}{1+\alpha\lambda} - \frac{4\eta_4}{1+\alpha\lambda}\right)\left(\frac{\alpha\eta_2}{1+\alpha\lambda} - T\right. \\
&\left. - \frac{V(k,1)R^2}{4}\right)\left[\frac{1}{12}\left(\rho\alpha R^2 + \frac{\eta_2k^2R^2}{1+\alpha\lambda}\right) + \frac{\eta_1}{1+\alpha\lambda}\right] \\
&+ k^2R^2\left[\rho\alpha^2R^2 + \frac{\alpha\eta_2k^2R^2}{1+\alpha\lambda} - V(k,1)R^2\right]\left[\frac{1}{24}\left(\rho\alpha R^2\right.\right. \\
&\left.\left. + \frac{\eta_2k^2R^2}{1+\alpha\lambda}\right) + \frac{\eta_1}{1+\alpha\lambda}\right]\left[\frac{1}{4}\left(\rho\alpha R^2 + \frac{\eta_2k^2R^2}{1+\alpha\lambda}\right) + \frac{2\eta_1}{1+\alpha\lambda}\right] = 0.
\end{aligned}$$

Here σ is the surface tension, and $K_n(\ )$ is the MacDonald function

$$V(k,n) = \frac{\rho_0(\alpha + ikU)^2}{kR}\frac{K_n(kR)}{K_n'(kR)} + \frac{\sigma}{R^3}(1 - n^2 - k^2R^2).$$

The effect of the shape of the rheological equation is manifested on the equation for the longitudinal stress T and the coefficients η_1, η_2, η_3, and η_4: the lower convective derivative $\eta_1 = \eta_2 = \eta/(1 - \lambda\Gamma)$, $\eta_3 = \eta_4 = \eta/(1 + 2\lambda\Gamma)$, $T = 3\eta\Gamma/[(1-\lambda\Gamma)(1 + 2\lambda\Gamma)]$, the upper convective derivative $\eta_1 = \eta_4 = \eta/(1 + \lambda\Gamma)$, $\eta_2 = \eta_3 = \eta/(1 - 2\lambda\Gamma)$, $T = 3\eta\Gamma/[(1 + \lambda\Gamma)(1 - 2\lambda\Gamma)]$, and the Jaumann derivative $\eta_1 = \eta_3 = \eta$, $\eta_2 = \eta(1 + (3/2)\lambda\Gamma)$, $\eta_4 = \eta(1 - (3/2)\lambda\Gamma)$, $T = 3\eta\Gamma$ (Γ is the velocity of primary extension).

In the long-wave approximation the equations derived coincide: with the Rayleigh equations [4] for an ideal liquid and account of capillary forces only; with the Bohr equation for stationary perturbations relative to a nozzle in a jet of a viscous liquid flowing from an elliptic nozzle (α = ikU); with the Weber equation [6] for axially symmetric perturbations of a viscous liquid jet; and with [2, 7] for axially symmetric perturbations of a viscoelastic jet with $\eta_1 = \eta_2 = \eta_3$. It was suggested in [6] to treat the bending oscillations of a viscous liquid jet by means of the theory of bending of an elastic beam. Unlike [6], the equations for bending perturbations were derived by us directly from the equations of hydrodynamics. Since the literature does not contain a detailed analysis of bending jet perturbations, we describe them in more detail. For $|\alpha\lambda| \ll 1$ the equations for viscous and viscoelastic liquids coincide. Retaining in the dispersion equation the dominant terms in kR, we obtain

$$\left(\alpha^2 - \frac{V(k,1)}{\rho}\right)\left[\frac{\alpha^2}{12} + \frac{4\alpha\eta}{3\rho R^2} + 4\left(\frac{\eta}{\rho R^2}\right)^2\right] + \frac{2\alpha\eta k^2}{\rho}\left[\frac{\alpha^2}{12}\frac{\alpha\eta}{\rho R^2} + \frac{3}{2}\left(\frac{\eta}{\rho R^2}\right)^2 k^2R^2\right] = 0. \tag{1}$$

When the viscous forces are small, the dispersion equation acquires the form

$$\alpha^2 + 2i\alpha\frac{\rho_0}{\rho+\rho_0}Uk - \frac{\left(\rho_0U^2 - \sigma/R\right)k^2}{\rho+\rho_0} = 0. \tag{2}$$

Two roots, corresponding to quickly damped perturbations, are lost in this case. A perturbation growth occurs for $\rho_0\rho U^2/(\rho+\rho_0) > \sigma/R$. We introduce the quantity $\xi = \left[\frac{\rho_0}{\rho+\rho_0}\left(1 - \frac{\sigma}{\rho_0U^2R}\right)\right]^{1/2}\cdot$ $\frac{\rho UR}{\eta}$, characterizing the relation between hydrodynamic and viscous forces. For $\xi \gg 1$ Eq. (2) is valid for all $kR \leqslant 1$. If $\xi \ll 1$, Eq. (2) applies only to sufficiently long waves $(\rho/(\rho + \rho_0))k^3R^3 \ll (8/3)\xi$. For shorter waves $(\rho/(\rho+\rho_0))k^3R^3 \gg (4/3)\xi$ the inertial forces are weak:

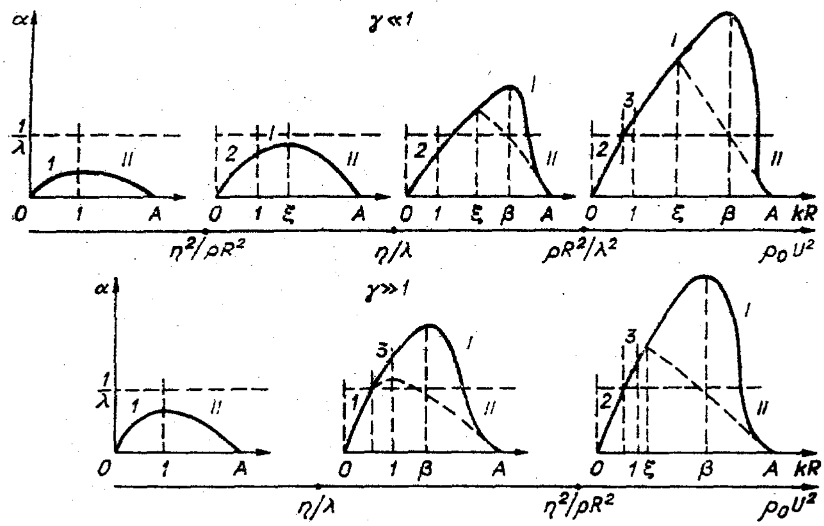

Fig. 1

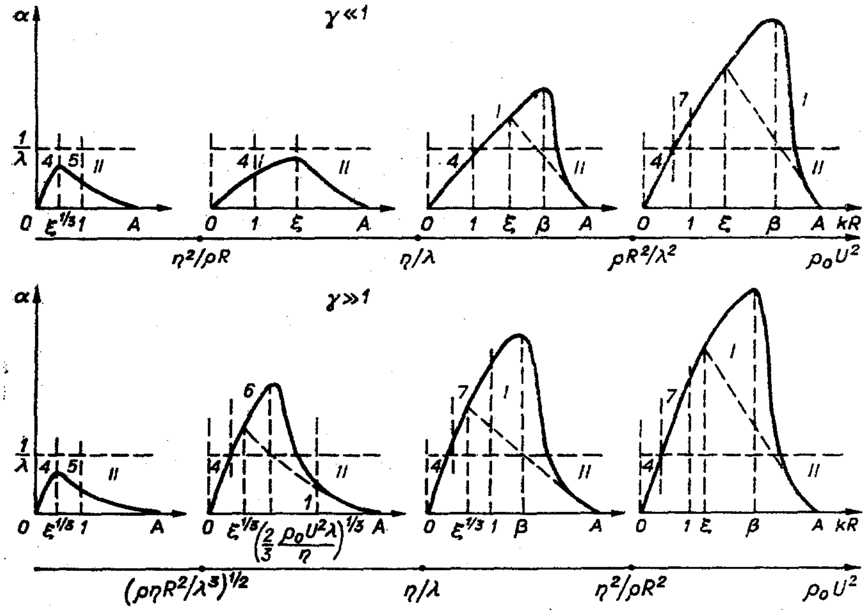

Fig. 2

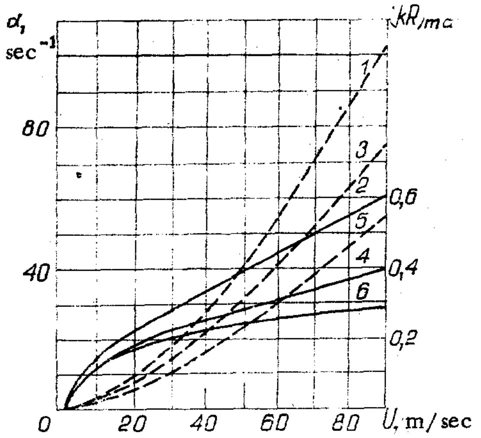

Fig. 3

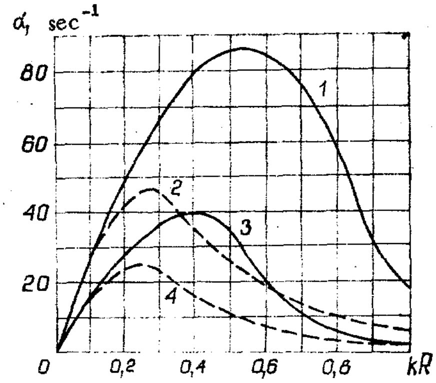

Fig. 4

$$\alpha \approx \frac{4\left(\rho_0 U^2 - \sigma/R\right)}{3\eta k^2 R^2\left[1 + i\frac{8\rho_0}{3\rho}\left(\frac{\rho U R}{\eta}\right)\frac{1}{k^3R^3}\right]}. \tag{3}$$

For $\xi \ll 1$ the depedence of the increment Re(α) on kR has a minimum at kR $\sim [((\rho + \rho_0)/\rho)\xi]^{1/3}$. The corresponding value is

$$\mathrm{Re}\,(\alpha)_{\max} \sim \left[\frac{\left(\rho_0 U^2 - \sigma/R\right)^2}{(\rho+\rho_0)\,\eta R^2}\right]^{1/3}.$$

We note that for $\left(\frac{\rho+\rho_0}{\rho}\xi^4\right)^{1/3} \ll 3$ one can replace Eq. (1) for all kR $\lesssim 1$ by

$$\alpha^2 + 2\alpha\left(i\frac{\rho_0}{\rho+\rho_0}Uk + \frac{3}{8}\frac{\eta k^4 R^2}{\rho+\rho_0}\right) - \frac{\left(\rho_0 U^2 - \sigma/R\right)k^2}{\rho+\rho_0} = 0. \tag{4}$$

An equation similar to (4), for $\rho_0 \ll \rho$, was derived by a different method in [8]. The same result is also obtained by the dispersion equation of [6], if it is taken into account that in this case $\alpha \ll 3\eta/\rho R^2$. The calculation of the velocity distribution across the jet gives for planar bending perturbations the expression

$$\mathbf{v} \approx \alpha\zeta_0\left[\left(1 - k^2\frac{x^2-y^2}{4}\right)\mathbf{e}_x - k^2\frac{xy}{2}\mathbf{e}_y - ikx\mathbf{e}_z\right],$$

similarly to the displacement distribution for planar bending of an elastic bar. Thus, the analogy suggested by Weber [6] between a jet and an elastic beam is valid for $(1 + \rho_0/\rho)^{1/3}\xi^{4/3} \ll 3$. If $\xi \gg 1$, the velocity distribution across the jet is given by the relation

$$\mathbf{v} \approx \alpha\zeta_0\left[\left(1 + k^2\frac{3x^2+y^2}{8}\right)\mathbf{e}_x + k^2\frac{xy}{4}\mathbf{e}_y + ikx\mathbf{e}_z\right],$$

and here the Weber analogy does not apply.

For $|\alpha\lambda| \gg 1$ the dispersion equation of bending oscillations is given in the following form, retaining only the main terms in kR:

$$\rho\alpha^2 - Tk^2\frac{\varkappa - \dfrac{4\eta_4}{\lambda\rho R^2}}{\varkappa + \dfrac{4\eta_4}{\lambda\rho R^2}} - V(k,1) + \frac{\eta_2 k^2}{\lambda}\frac{\dfrac{\varkappa^2}{6} + \dfrac{2\varkappa\eta_1}{\lambda\rho R^2} + \dfrac{2\eta_1\left(\eta_3 - \eta_2 + \dfrac{\eta_1}{2}\right)k^2R^2}{(\lambda\rho R^2)^2}}{\dfrac{\varkappa^2}{12} + \dfrac{\varkappa}{\lambda\rho R^2}\left(\eta_1 + \dfrac{\eta_4}{3}\right) + \dfrac{4\eta_1\eta_4}{(\lambda\rho R^2)^2}} = 0. \tag{5}$$

Here $\varkappa = \alpha^2 + (\eta_2/\lambda\rho)k^2$. For $|\varkappa| \ll 4\eta_4/\lambda\rho R^2$ Eq. (5) can be simplified:

$$\alpha^2 + 2i\alpha\frac{\rho_0}{\rho+\rho_0}Uk - \left(\rho_0 U^2 - \frac{\sigma}{R} - T - \frac{\eta_2\left(\eta_3 + \dfrac{\eta_1}{2}\right)}{2\lambda\eta_4}k^2R^2\right)\frac{k^2}{\rho+\rho_0} = 0. \tag{6}$$

The perturbation increases for

$$\frac{\rho\rho_0 U^2}{\rho+\rho_0} > \frac{\sigma}{R} + T + \frac{\eta_2\left(\eta_3 + \dfrac{\eta_1}{4}\right)}{2\lambda\eta_4}k^2R^2.$$

Thus, the primary extension increases the jet stability, and a compression lowers it. The increment values reach a maximum at

$$kR = \left[\frac{\lambda\eta_4\left(\dfrac{\rho\rho_0 U^2}{\rho+\rho_0} - \dfrac{\sigma}{R} - T\right)}{\eta_2\left(\eta_3 + \dfrac{\eta_1}{2}\right)}\right]^{1/2}, \quad \mathrm{Re}\,(\alpha)_{\max} = \left(\frac{\rho\rho_0 U^2}{\rho+\rho_0} - \frac{\sigma}{R} - T\right)\left[\frac{\lambda\eta_4}{2(\rho+\rho_0)R^2\eta_2\left(\eta_3 + \dfrac{\eta_1}{2}\right)}\right]^{1/2}$$

Equation (6) is valid for

$$\left[\frac{2(\rho+\rho_0)R^2\eta_2\left(\eta_3 + \dfrac{\eta_1}{2}\right)}{\lambda^3\eta_4}\right]^{1/2} \ll \rho_0 U^2 - \frac{\sigma}{R} - T \ll \left[\frac{8(\rho+\rho_0)\eta_2\left(\eta_3 + \dfrac{\eta_1}{2}\right)}{\lambda^2\rho}\right]^{1/2}$$

This inequality is possible only for $4\lambda\eta_4/\rho R^4 \gg 1$.

For $|\varkappa| \gg 4\eta_4/\lambda\rho R^2$ Eq. (5) can be written in the form

$$\alpha^2 + 2i\alpha\frac{\rho_0}{\rho+\rho_0}Uk - \left(\rho_0 U^2 - \frac{\sigma}{R} + T - \frac{2\eta_2}{\lambda}\right)\frac{k^2}{\rho+\rho_0} = 0. \tag{7}$$

The perturbations increase for $\rho\rho_0 U^2/(\rho + \rho_0) > \sigma/R + 2\eta_2/\lambda - T$. For all Maxwell liquids we have $(2\eta_2/\lambda - T) \sim \eta_2/\lambda$. Equation (7) is valid for

$$\rho_0 U^2 - \sigma/R + T - 2\eta_2/\lambda \gg \max\,[4\eta_4(\rho+\rho_0)/\lambda\rho,\ (\rho+\rho_0)R^2/\lambda^2].$$

It is seen from this inequality that the effect of elastic forces is insignificant in this case.

A calculation of the velocity field for a relaxing jet of a viscoelastic liquid shows that upon satisfaction of the conditions under which the dispersion equation of bending perturbation (5) reduces to (6), the analogy between a jet and a viscoelastic beam can be used. If the conditions under which (5) reduces to (7) are realized the analogy does not apply.

As is well known [4-6] for a viscous liquid, among the perturbations retaining jet linearity the axially symmetric ones increase most quickly. A similar result is also obtained for a viscoelastic liquid. For $|\alpha\lambda| \ll 1$ the dispersion equations coincide for viscous and viscoelastic liquids. Retaining only the main terms in kR, we obtain

$$(\rho + \rho_0 s)\,\alpha^2 + 2\alpha\left(i\rho_0 sUk + \frac{3}{2}\eta k^2\right) - \left[\rho_0 sU^2 + \frac{\sigma}{2R}(1 - k^2R^2)\right]k^2 = 0.$$

Here $s = -\frac{kR}{2}\frac{K_0(kR)}{K_0'(kR)} \approx -\frac{k^2R^2}{2}\ln\frac{kR}{2}$ for $kR \ll 1$. For

$$\left[\left(1 + \frac{\rho_0 s}{\rho}\right)\frac{\rho_0 s}{\rho}\left(\frac{\rho UR}{\eta}\right)^2 + \frac{\sigma R(\rho + \rho_0 s)}{2\eta^2}(1 - k^2R^2)\right] \gg 3kR \tag{8}$$

viscous forces can be neglected:

$$\alpha = -i\frac{\rho_0 s}{\rho+\rho_0 s}Uk \pm \left[\frac{\rho\rho_0 s}{\rho+\rho_0 s}U^2 + \frac{\sigma}{2R}(1 - k^2R^2)\right]^{1/2}\frac{k}{(\rho+\rho_0 s)^{1/2}}.$$

If the left-hand side of inequality (8) is significantly smaller than the right-hand side, the inertial terms do not have to be included:

$$\alpha = \frac{\rho_0 sU^2 + \frac{\sigma}{2R}(1 - k^2R^2)}{3\eta\left(1 + i\frac{2\rho_0 sU}{3\eta k}\right)}.$$

For $|\alpha\lambda| \gg 1$ we obtain from the dispersion equation for a viscoelastic liquid

$$\alpha = -i\frac{\rho_0 s}{\rho+\rho_0 s}Uk \pm \left[\frac{\rho\rho_0 s}{\rho+\rho_0 s}U^2 + \frac{T}{2} - \frac{\eta_1 + 2\eta_3}{\lambda} + \frac{\sigma}{2R}(1 - k^2R^2)\right]^{1/2}\frac{k}{(\rho+\rho_0 s)^{1/2}}.$$

The perturbations increase for

$$\frac{\rho\rho_0 s}{\rho+\rho_0 s}U^2 + \frac{\sigma}{2R}(1 - k^2R^2) > \frac{\eta_1 + 2\eta_3}{\lambda} - \frac{T}{2}.$$

This approximation is applicable if

$$\frac{\rho_0}{\rho_0 + 3\rho}\left(\frac{U\lambda}{R}\right)^2 + \frac{\sigma\lambda^2}{2R^3\left(\rho + \frac{\rho_0}{3}\right)} \gg 1 + \left(\frac{\eta_1 + 2\eta_3}{\lambda} - \frac{T}{2}\right)\frac{\lambda^2}{R^2\left(\rho + \frac{\rho_0}{3}\right)}$$

and if the following inequality is satisfied for sufficiently short waves:

$$kR \gtrsim \frac{(\rho + \rho_0 s)^{1/2}}{\left[\rho_0 s\left(\frac{U\lambda}{R}\right)^2 + \frac{\sigma\lambda^2}{2R^3}(1 - k^2R^2) - \frac{\lambda^2}{R^2}\left(\frac{\eta_1 + 2\eta_3}{\lambda} - \frac{T}{2}\right)\right]^{1/2}}.$$

For Maxwell liquids with lower convective, upper convective, and Jaumann derivatives, respectively, the quantity $(\eta_1 + 2\eta_3)/\lambda - T/2$ equals: $3\eta(1 - \lambda\Gamma/2)/[\lambda(1 - \lambda\Gamma)(1 + 2\lambda\Gamma)]$, $3\eta(1 - \lambda\Gamma/2)/[\lambda \times (1 + \lambda\Gamma)(1 - 2\lambda\Gamma)]$, $(3\eta/\lambda)(1 - \lambda\Gamma/2)$. As is seen from these expressions, the effect of primary extension or compression on axially symmetric perturbations reduces basically to an increase in the longitudinal viscosity.

Analysis of the dispersion equations for axially symmetric and bending perturbations shows the importance of the parameter $\gamma = (\eta_4\lambda/\rho R^2)^{1/2}$, for a viscoelastic liquid jet. This parameter equals the ratio of the distance traversed by a shear wave (propagating with velocity $(\eta/\rho\lambda)^{1/2}$[3]) during a relaxation time to the jet radius. A similar quantity, called the elasticity number, was derived in [9] by account of the dispersion equation of axially symmetric perturbations in a relaxing capillary jet of a Maxwell liquid, obtained in [9, 10]. The physical meaning of the parameter γ and its substantial effect on bending perturbations were, however, not considered.

Consider the breakup of a relaxing jet with Onesorg number $Z = \eta/\sqrt{\rho\sigma R} \gtrsim 1$, moving in air ($\rho \gg \rho_0$) with sufficiently high velocity $\rho_0 U^2 \gg \sigma/R$.

The analysis above is valid only for long-wave perturbations $kR \ll 1$. To describe the dynamics of short wave perturbations $\zeta_0 \exp[i(kz + n\varphi) + \alpha t]$, $n = 0 - 3$, $kR \gg 1$, one can use the dispersion equation for planar (since $kR \gg n$) perturbations on the surface of a semi-infinite layer of a relaxing viscoelastic liquid. The solution of this problem for a viscoelastic liquid is obtained from the corresponding solution for a viscous liquid (see [11]) with the replacement $\eta \to \eta/(1 + \alpha\lambda)$ [12]:

$$\left[\alpha + \frac{2\eta k^2}{\rho(1+\alpha\lambda)}\right]^2 = \frac{\rho_0 U^2 k^2 - \sigma k^3}{\rho} + \frac{4\eta^2 k^3}{\rho^2(1+\alpha\lambda)^2}\left[k^2 + \frac{\rho\alpha}{\eta}(1+\alpha\lambda)\right]^{1/2}. \tag{9}$$

In the case of a viscous liquid relation (9) can also be derived by the limiting transition $kR \gg 1$ in the exact dispersion equation for axially symmetric perturbations of a circular jet [11].

We analyze Eq. (9) in more detail than in [12]. For $|\alpha\lambda| \ll 1$ the equations for viscous and viscoelastic liquids coincide. For $Z \gtrsim 1$ and $1 \ll kR \ll \xi$ the viscous forces are weak:

$$\alpha \approx \sqrt{[(\rho_0 U^2 - \sigma k)/\rho]}\cdot k. \tag{10}$$

For $\xi \ll kR \lesssim \rho_0 U^2 R/\sigma$ the inertial forces are weak:

$$\alpha \approx (\rho_0 U^2 - \sigma k)/2\eta. \tag{11}$$

The kR-dependence of the maximum has a maximum at $kR \sim \xi$. The corresponding value is $\alpha_{max} \sim \rho_0 U^2/\eta$. For $\rho_0 U^2 \ll \eta/\lambda$ the effect of elastic forces is not manifested. For $\rho_0 U^2 \gg \eta/\lambda$ Eq. (9) also reduces to the two limiting expressions (10) and (11). This, however, occurs in a different range of wave numbers. Relation (10) is valid for $1 \ll kR \ll (\rho_0 U^2 - \eta/\lambda)/(\sigma/R)$, and (11) for $(\rho_0 U^2 - 2\eta/\lambda)/(\sigma/R) \lesssim kR \lesssim \rho_0 U^2 R/\sigma$. The kR-dependence of the increment has a maximun at $kR = \beta \approx 2\rho_0 U^2 R/(3\sigma)$. The corresponding value is $\alpha_{max} \approx (2/3\sqrt{3}))[(\rho_0 U^2)^{3/2}/(\sigma\sqrt{\rho})]$.

Figures 1 and 2 show schematically the dependence $\alpha(kR)$ for axially symmetric (Fig. 1) and bending (Fig. 2) perturbations for various values of $\rho_0 U^2$. The abscissa shows values of the aerodynamic pressure $\rho_0 U^2$ in the range relevant to the dependence $\alpha(kR)$. When there exists a portion $\alpha \gg 1/\lambda$ for a viscoelastic liquid, by the dashed lines we plot the portions of the curve $\alpha(kR)$ which differ for a Newtonian liquid with the same viscosity. Figures 1 and 2 illustrate the effect of viscoelastic forces on breakdown, generated by the aerodynamic action of air. The increments of axially symmetric perturbations for viscous and ideal liquids, as well as of bending perturbations for an ideal liquid, were calculated in [4, 14] without simplifying approximations. The corresponding curves for various values of the parameters were given in [15].

For curves 1 and 2 (Fig. 1) $\alpha\lambda \ll 1$, and the axially symmetric perturbations evolve as in the case of a viscous liquid. For curve 1, $\xi \ll 1$, and for $kR \gtrsim 2e^{-1/\xi^2}$ the inertial forces are weak: $\alpha \approx s\rho_0 U^2/3\eta$; for curve 2, $\xi \gg 1$, and the viscous forces are weak: $\alpha \approx \sqrt{\rho_0 s/\rho}\cdot Uk$. For curve 3, $\alpha\lambda \gg 1$, and the effect of elastic forces is manifested.* Here, however,

*Similar figures were constructed in [13] for long-wave breakup of a jet. However, maxima were erroneously noted on curves 1-3 for $kR \sim 1$. This is based on replacing the MacDonald functions by their first expansion terms for $kR \ll 1$. The remaining results of [13] are correct.

$\rho_0 U^2 \gg \eta/\lambda$, and they can be neglected. The dependence $\alpha(kR)$ of the long-wave breakup, 1–3, transform for $kR \gg 1$ to the corresponding dependences of the short-wave breakup, I and II.* Curve I describes expression (10), and curve II describes expression (11). The effect of elastic forces (the relation between $\rho_0 U^2$ and η/λ) in the short-wave region is mostly manifested in the wave number regions in which the approximate equations (10) and (11) are valid. It is seen from relation (9) (as well as (11)) that $\alpha(kR) = 0$ at $kR = \rho_0 U^2 R/\sigma$ (point A on Figs. 1 and 2). Perturbations with shorter waves are damped.

Comparison of the behavior of viscous and viscoelastic liquids shows that when $\gamma \ll 1$ and $\rho_0 U^2 \ll \eta/\lambda$ the breakup of a viscous and viscoelastic jet occurs identically. For $\rho_0 U^2 \ll \eta^2/\rho R^2$ the long-wave bending perturbations with wave numbers $kR \sim \xi^{1/3}$, $\alpha \sim (\rho_0 U^2/\sqrt{\rho \times \eta R^2})^{2/3}$. When $\rho_2/\rho R^2 \ll \rho_0 U^2 \ll \eta/\lambda$ the short-wave length perturbations $kR \sim \xi$, $\alpha \sim \rho_0 U^2/\eta$ develop more quickly. When $\gamma \gg 1$ the breakups of a viscous and a viscoelastic jet are similar only for $\rho_0 U^2 \ll \sqrt{\rho\eta R^2/\lambda^3}$. For $\sqrt{\rho\eta \times R^2/\lambda^3} \ll \rho_0 U^2 \ll \eta/\lambda$ long-wave perturbations $kR \sim (2\rho_0 U^2 \lambda/3\eta)^{1/2}$, $\alpha \sim \rho_0 U^2/3\eta\rho R^2/\lambda)^{1/2}$ increase more quickly in a viscoelastic jet. Their wavelength is shorter than for perturbations with the largest increment for a Newtonian liquid with the same viscosity. For $\rho_0 U^2 \gg \eta/\lambda$ short-wave perturbations $kR \sim 2\rho_0 U^2 R/3\sigma$, $\alpha \sim 2(\rho_0 U^2/3)^{3/2}/\sigma\sqrt{\rho}$ develop more quickly in a viscoelastic jet, independently of the quantity γ. In a Newtonian liquid with similar viscosity the perturbation wavelengths with maximum increment are larger.

Figure 3 shows the effect of jet velocity on the maximum value of the bending perturbation increment and the corresponding dimensionless wave number for a jet with parameters: $\rho = 1$ g/cm^3, $\rho_0 = 10^{-3}$ g/cm^3, $\eta = 10^3$ Pa·sec, $\sigma = 6 \cdot 10^{-2}$(N/m, $R = 10^{-2}$ m. For curves 1 and 2 $\lambda = 0.1$ sec, $\gamma = 31.6$) for curves 3 and 4 $\lambda = 0.03$ sec, $\gamma = 17.3$) curves 5 and 6 are a Newtonian liquid, $\gamma = 0$. The results of calculations, $\lambda = 10^{-3}$ sec, $\gamma = 3.16$ practically coincide with curves 5 and 6. As seen from Fig. 3, for low velocities the effect of elastic properties is insignificant. The maximum increment value increases with γ, and the wave number of the corresponding perturbation decreases. Figure 4 illustrates the effect of elastic properties on the nature of the dispersion curves. For curves 1 and 2 U = 80 m/sec, for 3 and 4 U = 50 m/sec. The odd numbers correspond to a viscoelastic liquid $\lambda = 0.1$ sec, and the even numbers to a Newtonian one. The remaining jet parameters are the same as in Fig. 3.

The effect of primary jet extension is manifested only on perturbations growing more quickly than the liquid relaxes.

For a jet with $\gamma \ll 1$ the effect of primary extension basically reduces to a change in the viscosity value. For $\gamma \gg 1$, along with a variation in the viscosity value the longitudinal tension decreases the effect of the aerodynamic pressure $\rho_0 U^2$ on the development of bending perturbations (see Eq. (6)).

Thus, the comparative analysis performed for viscous and viscoelastic liquids shows that the presence of elastic properties, described by the Maxwell model, changes qualitatively the nature of jet breakup.

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*For kR = 1 the expressions for α, given by the long-wave and short-wave approximations, differ only by numerical factors on the order of unity.

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